

# Decimal forms of rational numbers

This document explains why you always get either a terminating or a recurring decimal when you write a rational number as a decimal.

You can find the decimal form of a rational number by using long division to divide one positive integer into another. For example, you can find the decimal form of  $\frac{17}{25}$  by dividing 25 into 17, as shown below.

When you have carried out long division in the past, you might have used a layout in which the remainders are worked out underneath the main division, and the digits of the number that you are dividing into are brought down one by one. However, if you can work out the remainders in your head, or in some other way, then you can just write each remainder in front of the next digit along. This alternative layout is used here.

So here is how the decimal form of  $\frac{17}{25}$  can be worked out.

|                                        |                                   |
|----------------------------------------|-----------------------------------|
| Divide 25 into 17                      | $25 \overline{) 17.0000000\dots}$ |
| 25 into 1 goes 0 times, remainder 1    | $0$                               |
| 25 into 17 goes 0 times, remainder 17  | $25 \overline{) 17.0000000\dots}$ |
|                                        | $00$                              |
| 25 into 170 goes 6 times, remainder 20 | $25 \overline{) 17.1700000\dots}$ |
|                                        | $00.6$                            |
| 25 into 200 goes 8 times, remainder 0  | $25 \overline{) 17.1702000\dots}$ |
|                                        | $00.68$                           |

Since a remainder of zero has been reached, the long division stops, and the calculation shows that dividing 25 into 17 gives 0.68. That is, the decimal form of  $\frac{17}{25}$  is 0.68.

Here is another example. In this calculation the decimal form of  $\frac{5}{8}$  is found by dividing 8 into 5.

|                                     |                                 |
|-------------------------------------|---------------------------------|
| Divide 8 into 5                     | $8 \overline{) 5.0000000\dots}$ |
| 8 into 5 goes 0 times, remainder 5  | $0$                             |
| 8 into 50 goes 6 times, remainder 2 | $8 \overline{) 5.5000000\dots}$ |
|                                     | $0.6$                           |
| 8 into 20 goes 2 times, remainder 4 | $8 \overline{) 5.5020000\dots}$ |
|                                     | $0.62$                          |
| 8 into 40 goes 5 times, remainder 0 | $8 \overline{) 5.5025000\dots}$ |
|                                     | $0.625$                         |

Because a remainder of zero has been obtained, the long division stops, with the answer  $\frac{5}{8} = 0.625$ .

This applies even if the rational number is negative. For example, to find the decimal form of  $-\frac{5}{8}$ , divide 8 into 5 and put a minus sign in front of the answer.

In general, if you divide one positive integer into another, and you obtain a remainder of zero at some point in the long division, then the long division stops and you get a terminating decimal. You can see this happening in the next activity.

### Activity 1 *Finding a decimal form by long division*

Use long division to find the decimal form of the rational number  $\frac{3}{8}$ .

Now here is what happens when you divide 7 into 1, to find the decimal form of the rational number  $\frac{1}{7}$ . The first 20 steps are shown.

$$\begin{array}{r} 0.1428571428571428571\ldots \\ 7 \overline{) 1.10302060405010302060405010302060405010\ldots} \end{array}$$

No remainder of zero is obtained; instead the remainders are 1, 3, 2, 6, 4, 5, and then the remainder 1 is obtained again, so the pattern starts repeating. So

$$\frac{1}{7} = 0.\dot{1}42857.$$

In general, if you divide one positive integer into another, and you never get a remainder of zero, then eventually you will get a repeated remainder. This must happen because there are only so many possibilities for the remainders (each remainder is a non-negative integer that is smaller than the number you are dividing by). Once you obtain a repeated remainder, the pattern must repeat, so you get a recurring decimal. You should see this clearly if you do the next activity.

### Activity 2 *Finding more decimal forms by long division*

Use long division to find the decimal forms of the following numbers.

(a)  $\frac{1}{6}$       (b)  $\frac{2}{7}$

So when you divide one positive integer by another to find the decimal form of a rational number, there are two possibilities: either you get a remainder of zero in some step, or you do not. The first possibility gives a terminating decimal, while the second gives a recurring decimal.

## Solutions to activities

### Solution to Activity 1

We divide 8 into 3.

$$\begin{array}{r} 0.375 \\ 8 \overline{) 3.306040000\ldots} \end{array}$$

In the last step 8 is divided into 40, and this gives a remainder of zero, so the long division stops. So

$$\frac{3}{8} = 0.375.$$

### Solution to Activity 2

(a) We divide 6 into 1.

$$\begin{array}{r} 0.16666\ldots \\ 6 \overline{) 1.10404040\ldots} \end{array}$$

Each step from the third step onwards involves dividing 6 into 40 and obtaining 6 remainder 4, so the digit 6 repeats indefinitely. So

$$\frac{1}{6} = 0.1\dot{6}.$$

(b) We divide 7 into 2.

$$\begin{array}{r} 0.2857142857142\ldots \\ 7 \overline{) 2.20604050103020604050103020\ldots} \end{array}$$

We obtain

$$\frac{2}{7} = 0.\dot{2}8571\dot{4}.$$